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Non-Schlesinger deformations of ordinary differential equations with rational coefficients

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Abstract

We consider deformations of 2×2 and 3×3 matrix linear ODEs with rational coefficients with respect to singular points of Fuchsian type which do not satisfy the well known system of Schlesinger equations (or its natural generalization). Some general statements concerning the reducibility of such deformations for 2×2 ODEs are proved. An explicit example of the general non-Schlesinger deformation of 2×2-matrix ODEs of the Fuchsian type with four singular points is constructed and application of such deformations to the construction of special solutions of the corresponding Schlesinger systems is discussed. Some examples of isomonodromic and non-isomonodromic deformations of 3×3 matrix ODEs are considered. The latter arise as the compatibility conditions with linear ODEs with non-single-valued coefficients.

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1. Introduction

It is well known that isomonodromic deformations of the Fuchsian matrix ODE,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1} + \frac{A_t}{\lambda - t}\right)\Psi \tag{1.1}$$

where $A_\infty = A_0 + A_1 + A_t$ is normalized to be independent of t, are governed by the system of Schlesinger equations [1],

$$\frac{\mathrm{d}A_0}{\mathrm{d}t} = \frac{1}{t}[A_t,\ A_0] \qquad \frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{1}{t-1}[A_t,\ A_1] \qquad \frac{\mathrm{d}A_t}{\mathrm{d}t} = \left[\frac{1}{t}A_0 + \frac{1}{t-1}A_1,\ A_t\right]. \tag{1.2}$$

This system is the compatibility condition of equation (1.1) with the following ODE:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\frac{A_t}{\lambda - t}\Psi. \tag{1.3}$$

Throughout this paper we consider matrix ODEs with respect to independent variable λ whose coefficients are rational functions of λ with at least one Fuchsian singularity at $\lambda = t$. We call deformations of such ODEs Schlesinger (with respect to t) if these ODEs are compatible with equation (1.3). This is a natural generalization of the notion of Schlesinger deformations of the Fuchsian ODEs. It is clear that the Schlesinger deformations are the simplest isomonodromic deformations. On the other hand, as follows from the work by Malgrange [2], any isomonodromic deformation of the Fuchsian system could be described *up to isomorphism* by the Schlesinger deformations. The purpose of this paper is to understand better what 'up to isomorphism' means from the point of view of the theory of integrable systems and special functions of the isomonodromy type [3]. More precisely, the question is whether this isomorphism can be always constructed explicitly, or in other words can we obtain any new integrable systems or special functions by considering non-Schlesinger deformations of ODEs with rational coefficients?

The results presented in sections 2 and 3 give a negative answer to this question in the case of the isomonodromic deformations: any integrable system which describes non-Schlesinger isomonodromic deformations of any Fuchsian ODE (or weak non-Schlesinger (see section 2) isomonodromic deformation for non-Fuchsian ODEs with rational coefficients) can be mapped explicitly by a proper Schlesinger *transformation* into some integrable system describing deformation of the Schlesinger type. This statement can be reformulated in the following way: there are no new transcendental functions defined via non-Schlesinger isomonodromic deformations of the Fuchsian ODEs. The sketch of this rather simple proof is given at the end of section 3. It was communicated to me by Bolibruch for the case of the Fuchsian ODEs a few weeks after the workshop where the results of this paper were reported. Moreover, he pointed out to me his recent work [4], where he also studied non-Schlesinger deformations of the Fuchsian ODEs but looking at them from a different angle: the isomonodromy confluences of the Fuchsian singularities. However, Bolibruch did not study the reducibility of the ODEs subject to non-Schlesinger deformations, which I address, in particular, in this paper. The latter means further simplification of these ODEs. Moreover, such reducibility does not follow from the proof mentioned above and has a remarkable consequence for construction of the explicit solutions of the Painlevé and higher Painlevé equations, and (generalized) Garnier type systems.

In section 4 some simplest examples of non-isomonodromic deformations of 3×3 matrix ODEs, which are, of course, non-Schlesinger, are presented. Here for simplicity we consider non-Fuchsian type ODEs since one of our tasks here is to show that non-isomonodromic deformations, which can be described via compatibility conditions (generalized Schlesinger equations), really exist. For the Fuchsian ODEs one can consider analogous examples. In our examples, presented in section 4, there appear only elementary and Painlevé type functions. The latter non-evident fact was established (also after the workshop) by Cosgrove. However, to the best of my knowledge no general statements concerning reducibility of such deformations to those of isomonodromy type are known, so further interesting studies of such deformations are anticipated.

2. Non-Schlesinger isomonodromic deformations of 2×2 matrix ODEs

We begin our study with the special case of equation (1.1) when $A_k \in sl_2(\mathbb{C})$ for $k = 0, 1, t$ and ∞. Let us suppose that $A_\infty = -\frac{\theta_\infty}{2}\sigma_3$, $\theta_\infty \neq 0$, which is in fact (excluding one exceptional solvable case) also a normalization rather than a restriction on A_k. In this case the system of Schlesinger equations (1.2) is equivalent to the sixth Painlevé equation [5].

Equation (1.3) is a sufficient condition that monodromy matrices of a fundamental solution of equation (1.1) are independent of t. In the general situation this condition is also necessary, but there are some special exceptional cases, which are discussed here. A possibility of such deformations resulted from the non-uniqueness of solubility of the inverse monodromy problem for some special monodromy matrices.

We begin with a simple generalization of equation (1.3),

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\frac{A_t + \Lambda}{\lambda - t}\Psi. \tag{2.1}$$

The compatibility condition of equations (1.1) and (2.1) reads

$$[\Lambda,\ A_t] = \Lambda \tag{2.2}$$

$$\begin{aligned} \frac{\mathrm{d}A_0}{\mathrm{d}t} &= \frac{1}{t}[A_t + \Lambda,\ A_0] \qquad \frac{\mathrm{d}A_1}{\mathrm{d}t} = \frac{1}{t-1}[A_t + \Lambda,\ A_1] \\ \frac{\mathrm{d}A_t}{\mathrm{d}t} &= \left[\frac{1}{t}A_0 + \frac{1}{t-1}A_1,\ A_t + \Lambda\right]. \end{aligned} \tag{2.3}$$

Equation (2.2) implies $\operatorname{tr}\Lambda = 0$, $\Lambda^2 = 0$, and $\pm 1/2$ are the eigenvalues of A_t. Denote

$$A_t = \begin{pmatrix} a_t & b_t \\ c_t & -a_t \end{pmatrix}$$

then the solution of equation (2.2) can be written as follows:

$$\Lambda = \mu \begin{pmatrix} -(a_t + 1/2)b_t & -b_t^2 \\ (a_t + 1/2)^2 & (a_t + 1/2)b_t \end{pmatrix} \qquad \text{for} \quad a_t \neq -1/2 \tag{2.4}$$

and

$$\Lambda = \mu \begin{pmatrix} c_t & 1 \\ -c_t^2 & -c_t \end{pmatrix} \qquad \text{for} \quad a_t = -1/2 \tag{2.5}$$

where $\mu = \mu(t)$ is an arbitrary function of t.

Consider now the asymptotic expansion of a fundamental solution of equation (1.1) as $\lambda \to t$,

$$\Psi = \sum_{k=0}^{\infty} \psi_k (\lambda - t)^k (\lambda - t)^{\sigma_3/2} \begin{pmatrix} 1 & \kappa \ln(\lambda - t) \\ 0 & 1 \end{pmatrix} C \qquad \det C = \det \psi_0 = 1 \tag{2.6}$$

where the parameter $\kappa = 0$ or 1. One finds

$$A_t = \tfrac{1}{2}\psi_0 \sigma_3 \psi_0^{-1} \qquad \Lambda = (C_{21}'C_{22} - C_{22}'C_{21})\psi_0 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \psi_0^{-1}$$

where C_{21} and C_{22} are the corresponding matrix elements of C and the primes denotes differentiation on t. The monodromy matrix of Ψ corresponding to any loop, with only one singular point $\lambda = t$ inside, reads

$$M_t = -I - 2\pi \mathrm{i}\kappa \begin{pmatrix} C_{21}C_{22} & C_{22}^2 \\ -C_{21}^2 & -C_{21}C_{22} \end{pmatrix}.$$

If $\kappa \neq 0$ then the isomonodromy condition, $\frac{\mathrm{d}}{\mathrm{d}t}M_t = 0$, implies $C_{21}'C_{22} - C_{22}'C_{21} = 0$ and therefore $\Lambda = 0$. Thus, in the case of 2×2 matrices non-Schlesinger deformations of the type (2.1) exist if and only if $\kappa = 0$. This condition can be written in terms of equation (1.1) as follows:

$$\operatorname{tr}\left(\left(\frac{1}{t}A_0 + \frac{1}{t-1}A_1\right)\left(A_t + \frac{1}{2}\sigma_3\right)\right) = 0. \tag{2.7}$$

Consider the explicit construction of this deformation. Taking into account the integrals of the Schlesinger system (1.2),

$$A_t = -A_0 - A_1 - \frac{\theta_\infty}{2}\sigma_3 \qquad \operatorname{tr} A_k = 0 \qquad \det A_k = -\frac{\theta_k^2}{4}$$

where $\theta_t = 1$ and θ_k, $k = 0, 1$ and ∞ are the constants of integration considered as parameters; we see that the most general non-Schlesinger deformation of the type (2.1) depends on the function $\mu(t)$ (see equations (2.4) and (2.5)) and five parameters (constants of integration of equations (2.3)): θ_0, θ_1, θ_∞ and c_1, c_2. One of them can be introduced via the gauge transformation,

$$A_k \longrightarrow c_2^{\sigma_3} A_k c_2^{-\sigma_3}. \tag{2.8}$$

Due to the absence of the log-term in the expansion equation (2.6), the singularity at $\lambda = t$ in equation (1.1) is removable via a proper Schlesinger transformation. Let us describe this construction in detail. Consider the hypergeometric equation, which can be written in matrix form as follows:

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\lambda} = \left(\frac{A}{\lambda} - \frac{\frac{\theta_\infty+1}{2}\sigma_3 + A}{\lambda - 1}\right)\Phi \qquad A = \begin{pmatrix} -\frac{\theta_0^2-\theta_1^2+(\theta_\infty+1)^2}{4(\theta_\infty+1)} & \frac{\theta_0^2-(\theta_\infty+1-\theta_1)^2}{4(\theta_\infty+1)} \\ -\frac{\theta_0^2-(\theta_\infty+1+\theta_1)^2}{4(\theta_\infty+1)} & \frac{\theta_0^2-\theta_1^2+(\theta_\infty+1)^2}{4(\theta_\infty+1)} \end{pmatrix}. \tag{2.9}$$

Note that $\det A = -\frac{\theta_0^2}{4}$ and $\det(\frac{\theta_\infty+1}{2}\sigma_3 + A) = -\frac{\theta_1^2}{4}$. An explicit formula for the fundamental solution $\Phi = \Phi(\lambda)$ of this equation in terms of the Gauß hypergeometric functions can be found for example in [6]. In the following we denote as Φ any fundamental solution of equation (2.9) which is independent of t. A fundamental solution of the system (1.1), (1.3), which corresponds to the general (modulo gauge transformation (2.8)) solution of the system (2.1), (1.1) (for $\theta_t = 1$) can be written as follows:

$$\Psi = D^{-1}R^{-1}(\lambda, t)\Phi \qquad D = \begin{pmatrix} 1 & 1 \\ 0 & \kappa_1 \end{pmatrix} \qquad \kappa_1 = \frac{4\theta_\infty(\theta_\infty + 1)}{(\theta_\infty + 1 - \theta_1)^2 - \theta_0^2} \tag{2.10}$$

$$R(\lambda, t) = \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\sqrt{\lambda - t} + \begin{pmatrix} 1 & 0 \\ c_1(t) & 0 \end{pmatrix}\frac{1}{\sqrt{\lambda - t}}\right) \qquad \det R(\lambda, t) = 1 \tag{2.11}$$

where $c_1(t)$ is an arbitrary function of t and for both square roots in equation (2.11) the same branch should be chosen. The corresponding solution of the system (2.2), (2.3) reads

$$\begin{aligned} &A_0 = D^{-1}R^{-1}(0, t)ADR(0, t) \qquad && A_1 = -D^{-1}R^{-1}(1, t)\left(\frac{\theta_\infty + 1}{2}\sigma_3 + A\right)DR(1, t) \\ &A_t = D^{-1}\begin{pmatrix} 1/2 & 0 \\ r(t) & -1/2 \end{pmatrix}D \qquad && \Lambda = D^{-1}\begin{pmatrix} 0 & 0 \\ \frac{\mathrm{d}c_1(t)}{\mathrm{d}t} - r(t) & 0 \end{pmatrix}D \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} r(t) = {} & ((\theta_0^2 - (\theta_\infty + 1 - \theta_1)^2)c_1^2(t) + (4t(\theta_\infty + 1)^2 - 2((\theta_\infty + 1)^2 \\ & + \theta_0^2 - \theta_1^2))c_1(t) + \theta_0^2 - (\theta_\infty + 1 + \theta_1)^2)\{4t(t - 1)(\theta_\infty + 1)\}^{-1}. \end{aligned} \tag{2.13}$$

The function $c_1(t)$ depends on the function $\mu(t)$ and the parameter c_1, which is introduced in the paragraph above equation (2.8). Actually, if the function $\mu(t)$ is given, as one of the coefficients in the system (2.2), (2.3), then

$$\mu(t) = \frac{\kappa_1(c_1'(t) - r(t))}{(r(t) - \kappa_1)^2}. \tag{2.14}$$

The function $c_1(t)$ is thus the general solution of the differential equation (2.14) and, therefore, depends on the constant of integration c_1.

At this stage it is worthwhile to note an application of the non-Schlesinger deformations to the construction of one-parameter families of solutions of the sixth Painlevé equation. As mentioned above, the sixth Painlevé equation corresponds to the Schlesinger deformation (1.3) of equation (1.1), i.e. $\mu(t) \equiv 0 \Longrightarrow c_1'(t) = r(t)$. The latter is nothing but the Riccati equation, whose solution can be written in terms of the logarithmic derivative of the Gauß hypergeometric function. An explicit formula for the solution of the sixth Painlevé equation is than easy to obtain from the equations (2.12) and a parametrization of the solution of the sixth Painlevé equation in terms of matrix elements of A_k given in [5]. This solution corresponds to $\theta_t = \pm 1$.

It is important to notice that there is another construction of the one-parameter families of the solutions of the sixth Painlevé equation; it is based on on the 'triangular reduction' of the system (1.1), (1.3) (the latter also leads to the hypergeometric functions, but for $\theta_0 + \theta_1 + \theta_t + \theta_\infty = 0$, see, e.g., [3]). These two constructions give, modulo application of the Schlesinger transformations and fractional-linear transformations of λ, all one-parameter families of solutions of the sixth Painlevé equation. The fact that there are no other solutions of the sixth Painlevé equation follows from the work [7]. It is clear that application of the non-Schlesinger deformations to the construction of the special solutions of the Garnier systems and so-called higher Painlevé equations should 'break the symmetry' with the construction of the special solutions based on the triangular reduction of the associated linear ODEs. The latter always leads to the linear ODEs for (generally multivariable) hypergeometric functions, whilst the construction related to the non-Schlesinger deformations leads to the Riccati equations (for 2×2 matrix ODEs) which can be transformed to the linear ODEs of the second order whose coefficients are defined by solutions of the Garnier systems (Painlevé/higher Painlevé equations), which are junior members of the corresponding hierarchies.

The most general non-Schlesinger isomonodromic deformations of equation (1.1) are defined by the compatibility condition of equation (1.1) with the following ODE:

$$\frac{\partial \Psi}{\partial t} = \left(-\frac{A_t + \Lambda_1^t}{(\lambda - t)} + \sum_{k=2}^{n_t} \frac{\Lambda_k^t}{(\lambda - t)^k} + \sum_{k=1}^{n_1} \frac{\Lambda_k^1}{(\lambda - 1)^k} + \sum_{k=1}^{n_0} \frac{\Lambda_k^0}{\lambda^k} \right) \Psi \tag{2.15}$$

where $n_t \geqslant 1$, $n_1 \geqslant 0$ and $n_0 \geqslant 0$ are integers. The notation is chosen such that if $\Lambda_{n_p}^p = 0$, then $\Lambda_k^p = 0$ for all $k \leqslant n_p$, and if the upper limit of the sum is less than its lower limit, then the sum is absent. One proves that $\operatorname{tr} \Lambda_k^p = 0$ and $(\Lambda_{n_p}^p)^2 = 0$ for $p = t$, 1 and 0. Moreover,

$$\Lambda_{n_p}^p \neq 0 \Longrightarrow A_p = \frac{n_p}{2} \psi_0^p \sigma_3 (\psi_0^p)^{-1} \qquad \Lambda_{n_p}^p = \mu_{l_p}^p(t) \psi_0^p \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (\psi_0^p)^{-1} \tag{2.16}$$

the corresponding monodromy matrix at $\lambda = p$ is $M_p = (-1)^{n_p}$; this means that the function Ψ has the following expansion at $\lambda = p$:

$$\Psi = \sum_{k=0}^{\infty} \psi_k (\lambda - p)^k (\lambda - p)^{\frac{n_p}{2}\sigma_3} C \qquad \det C = \det \psi_0 = 1. \tag{2.17}$$

It follows from this expansion that the singularity at $\lambda = p$ can be removed from equations (1.1) and (2.15) via n_p transformations of the type (2.10). In general, the non-Schlesinger isomonodromic deformation defined by equation (2.15) can be parametrized via $n_p + n_1 + n_0$ arbitrary functions of t. More precisely: if among the 'senior' matrices $\Lambda_{n_k}^k$, $k = 0$, 1 and t, only one matrix $\Lambda_{n_p}^p$ is different from 0, then via n_p Schlesinger transformations of the form (2.10) and, possibly (if $p \neq t$), by permutation of the points 0, 1 and t, one transforms Ψ, the fundamental solution of the system (1.1), (2.15), to the solution of the hypergeometric equation (2.9) Φ. If any two of the 'senior' matrices $\Lambda_{n_p}^p \neq 0$ and $\Lambda_{n_q}^q \neq 0$, then $n_p + n_q$ Schlesinger transformations convert Ψ to the the function $\Phi = (\lambda - r)^{-(\theta_\infty + 1)\sigma_3/2}$, where $r \neq p, q$ and $r \in \{0, 1, t\}$, so that in particular $\theta_r = -\theta_\infty - 1$. Finally, if all three matrices

$\Lambda^k_{n_k} \neq 0$, $k = 0, 1$ and t, then Ψ can be presented as a multiplication of the $n_t + n_0 + n_1$ Schlesinger transformations.

Since the consideration above is quite local this result can be generalized for 2×2 matrix ODEs

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = A(\lambda)\Psi \tag{2.18}$$

where $= A(\lambda) \in sl_2(\mathbb{C})$ is an arbitrary rational function of λ.

Definition. *Let t be a pole of $A(\lambda)$ of the first order with the residue A_t,*

$$A(\lambda) = B(\lambda) + \frac{A_t}{\lambda - t} \tag{2.19}$$

where $B(\lambda)$ has no pole at $\lambda = t$ as a rational function of λ and A_t is independent of λ. If a fundamental solution of equation (2.18) satisfies also equation (1.3), then we call any solution of the system representing the compatibility condition of equations (1.3), (2.18) and (2.19), namely,

$$\frac{\partial B(\lambda)}{\partial t} = \left[\frac{B(\lambda) - B(t)}{\lambda - t}, A_t\right] \qquad \frac{\mathrm{d}A_t}{\mathrm{d}t} = [B(t), A_t]$$

the Schlesinger deformation of equation (2.18) with respect to the parameter t.

Remark 2.1. The fourth, fifth and sixth Painlevé equations can be written as the Schlesinger deformations of 2×2 matrix linear ODEs.

Definition. *Denote by t_1 any other first-order pole of the rational function $A(\lambda)$; in particular, it may coincide with t. We say that isomonodromic deformation with respect to t is t_1-non-Schlesinger iff the rational function $R(\lambda)$ in the equation*

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \left(-\frac{A_t}{\lambda - t} + R(\lambda)\right)\Psi \tag{2.20}$$

has a pole at $\lambda = t_1$.

Proposition 2.1. *Any t_1-non-Schlesinger deformation of equation (2.18) with respect to t can be transformed, by a finite number of the Schlesinger transformations of the type given by the first equation (2.10) where $t \to t_1$ and the matrix D is independent of λ, to the ODE of the form*

$$\frac{\mathrm{d}\hat{\Phi}}{\mathrm{d}\lambda} = \hat{A}(\lambda)\hat{\Phi} \tag{2.21}$$

where the rational function $\hat{A}(\lambda) \in sl_2(\mathbb{C})$ has no pole at $\lambda = t_1$ whilst its other poles and their orders coincide with those of the function $A(\lambda)$.

Moreover, if, additionally, the function $A(\lambda)$ is isomonodromic with respect to t_1, namely,

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t_1} = \left(-\frac{A_{t_1}}{\lambda - t_1} + R_1(\lambda)\right)\Psi \tag{2.22}$$

and the set of poles ($\in\mathbb{CP}^1$) of the rational function $R_1(\lambda)$ coincides with a subset ($\in\mathbb{CP}^1$) of the first-order poles of the function $A(\lambda)$, then, possibly by applying a finite number of additional Schlesinger transformations of the type described in the previous paragraph, one arrives at equation (2.21), where the function $\hat{A}(\lambda)$ is independent of t_1 (note that if $t_1 = t$, then $R(\lambda) = R_1(\lambda)$ has a pole at $\lambda = t$). The set of poles of $\hat{A}(\lambda)$ is a subset of the poles of $A(\lambda)$.

Remark 2.2. We call the non-Schlesinger deformations described in the second paragraph of proposition 2.1 weak non-Schlesinger deformations. If one allows that among the poles of $R_1(\lambda)$ there are some non-Fuchsian singularities of equation (2.18), then such isomonodromic deformations could be called strong non-Schlesinger deformations. For the strong non-Schlesinger deformations the latter statement of proposition 2.1, in general, is not true.

If we suppose that equation (2.18) suffers from some weak non-Schlesinger deformation with respect to t and $R(\lambda)$ (see equation (2.20)) has no pole at $\lambda = t$, then, by applying proper Schlesinger transformations, equation (2.18) can be simplified to equation (2.21) with the matrix $\hat{A}(\lambda)$ having fewer poles than $A(\lambda)$ and the Schlesinger t-dependence.

3. Non-Schlesinger isomonodromic deformations of 3×3 matrix ODEs

In the case when the 2×2 matrix ODE (2.18) with the connection matrix (2.19) suffers from weak non-Schlesinger isomonodromic deformations with respect to t, it is always reducible via the Schlesinger transformations to a simpler ODE with a less number of poles; this is not always the case for the ODEs of higher matrix dimension. For example, one can take a direct sum of two 2×2 equations of the type (1.1): coefficients of the first equation are deformed by the Schlesinger deformation (1.3), whilst coefficients of the second one suffer from a non-Schlesinger deformation of the type (2.1). Clearly, we have constructed a non-Schlesinger isomonodromic deformation of the 4×4 ODE with respect to t which (in the general situation) cannot be reduced via any Schlesinger transformation to any equation independent of t. However, it can be transformed (via an appropriate Schlesinger transformation) to the equation for which deformation with respect to t is of the Schlesinger type, so the natural question is whether there are any non-Schlesinger isomonodromic deformations of equation (2.18) which are not reducible via Schlesinger transformations to the Schlesinger deformations of some other ODE of the same type (2.18).

Consider the simplest non-Schlesinger isomonodromic deformation of equations (2.18) and (2.19) of the type (2.1) for the case of 3×3 matrices. Equation (2.2) now implies $\operatorname{tr}\Lambda = 0$ and $\Lambda^3 = 0$. Using this one finds that the only possible solutions of equation (2.2) read

$$(1)\ A_t = \psi_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ \mu_1(t) & 0 & 0 \\ 0 & \mu_2(t) & 0 \end{pmatrix} \psi_0^{-1} \tag{3.1}$$

$$(2)\ A_t = \psi_0 \begin{pmatrix} 1/3 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \mu_1(t) & \mu_2(t) & 0 \end{pmatrix} \psi_0^{-1} \tag{3.2}$$

$$(3)\ A_t = \psi_0 \begin{pmatrix} 1/3 & 1 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu_2(t) & 0 \end{pmatrix} \psi_0^{-1} \tag{3.3}$$

$$(4)\ A_t = \psi_0 \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ \mu_1(t) & 0 & 0 \\ \mu_2(t) & 0 & 0 \end{pmatrix} \psi_0^{-1} \tag{3.4}$$

$$(5)\ A_t = \psi_0 \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 1 \\ 0 & 0 & -1/3 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ \mu_1(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \psi_0^{-1}. \tag{3.5}$$

The matrix ψ_0 is a function of t with $\det \psi_0 = 1$; $\mu_1(t)$ and $\mu_2(t)$ are some functions of t. Actually, from the system (2.3) it follows that for any choice of the functions $\mu_k(t)$ there is a solution of the system (2.3). This means that the general solution in each case depends on two (cases 1, 2 and 4) or one (cases 3 and 5) arbitrary functions of t. The functions $\mu_1(t)$ and $\mu_2(t)$ may also depend on some other pole parameters (if any).

Consider the first case. Any fundamental solution at $\lambda = t$ has the following asymptotic expansion:

$$\Psi \underset{\lambda\to t}{=} \sum_{k=0}^{\infty} \psi_k(\lambda - t)^k \begin{pmatrix} \lambda - t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/(\lambda - t) \end{pmatrix} (\lambda - t)^{\Xi} C \tag{3.6}$$

$$\Xi = \begin{pmatrix} 0 & \kappa_2 & \kappa_1 \\ 0 & 0 & \kappa_3 \\ 0 & 0 & 0 \end{pmatrix} \qquad \det C = \det \psi_0 = 1 \tag{3.7}$$

where κ_l, $l = 1, 2, 3$, are parameters independent of t. Denote

$$Q = C'(t)C^{-1}(t) \qquad \Rightarrow \operatorname{tr} Q = 0.$$

Now using expansion (3.6) one proves that

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \left(\frac{\Lambda_2}{(\lambda - t)^2} - \frac{A_t + \Lambda}{\lambda - t} \right) \Psi \tag{3.8}$$

$$\Lambda_2 = \psi_0 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ Q_{31} & 0 & 0 \end{pmatrix} \psi_0^{-1} \qquad \Lambda = -\psi_0 \begin{pmatrix} 0 & 0 & 0 \\ Q_{21} & 0 & 0 \\ 0 & Q_{32} & 0 \end{pmatrix} \psi_0^{-1} \tag{3.9}$$

where Q_{ik} are matrix elements of Q and A_t is given by equation (3.1). In fact, equation (2.1) is a special case of equation (3.8) corresponding to the case $Q_{31} = 0$. The monodromy matrix of the fundamental solution with expansion (3.6) corresponding to a small loop around $\lambda = t$ reads

$$M_t = C^{-1} \mathrm{e}^{2\pi \mathrm{i} \Xi} C$$

so the isomonodromy condition, $\frac{\mathrm{d}}{\mathrm{d}t} M_t = 0$, is equivalent to

$$[Q, \mathrm{e}^{2\pi \mathrm{i} \Xi}] = 0. \tag{3.10}$$

The result of the analysis of equation (3.10) can be formulated as the following

Proposition 3.1. *In a generic situation, i.e.* $\nu(t) = Q_{31} \neq 0$ *or* $\nu(t) = Q_{31} = 0$ *but* $\mu_1(t)\mu_2(t) = Q_{21}Q_{32} \neq 0$, *the isomonodromy condition (3.10) implies* $\Xi = 0$; *therefore, there is a Schlesinger transformation which transfers equation (2.18) with the Fuchsian singularity at* $\lambda = t$ *into an equation of the same type but without this singularity.*

If $\nu(t) = 0$, $\mu_2(t) = 0$ *but* $\mu_1(t) \neq 0$, *then* $\kappa_2 = 0$. *If* $\nu(t) = 0$, $\mu_1(t) = 0$ *but* $\mu_2(t) \neq 0$, *then* $\kappa_3 = 0$.

Remark 3.1. In each of the two last cases of proposition 3.1 there are some further necessary conditions on Q, following from equation (3.10), which allow us to have $\Xi \neq 0$. The question is whether these conditions are sufficient, namely, can one construct a non-Schlesinger isomonodromic deformation of the type (2.1) for some ODE of the form (2.18) such that equation (3.10) is satisfied and the matrix $\Xi \neq 0$, is opened?

The analysis of cases 2 and 4 (see equations (3.2) and (3.4)) essentially repeats the previous one. Some minor modifications, say, for case 2 are as follows. Matrix $\Lambda_2 = 0$, $\mu_1(t) = Q_{31}$ and $\mu_2(t) = Q_{32}$. Instead of expansion (3.6) we have

$$\Psi \underset{\lambda\to t}{=} \sum_{k=0}^{\infty} \psi_k(\lambda - t)^k \begin{pmatrix} (\lambda - t)^{\frac{1}{3}} & 0 & 0 \\ 0 & (\lambda - t)^{\frac{1}{3}} & 0 \\ 0 & 0 & (\lambda - t)^{-\frac{2}{3}} \end{pmatrix} (\lambda - t)^{\Xi} C \tag{3.11}$$

where the matrices Ξ and C have the same properties as those in (3.7). The monodromy matrix around $\lambda = t$ reads $M_t = e^{2\pi i/3} C^{-1} e^{\Xi} C$, therefore the isomonodromy condition again reduces to equation (3.10), so the part of proposition 3.1 for $\nu(t) = 0$ and remark 3.1 apply also for cases 2 and 4 without any modifications.

Minor modifications are required also in the remaining cases 3 and 5 (see equations (3.3) and (3.5)). In case 3 expansion of the function Ψ at $\lambda = t$ is just a special case of the expansion corresponding to case 2 (3.11) in which $\kappa_1 = 0$ and $\kappa_2 = 1$ see [8]. The non-Schlesinger condition $\mu_2(t) = Q_{32} \neq 0$ implies $\kappa_3 = 0$. Similarly, the non-Schlesinger condition in case 5, i.e., $\mu_1(t) = 0$, leads to the condition $\kappa_1 = 0$, $\kappa_2 = 0$ and $\kappa_3 = 1$. In these cases the log-terms in the corresponding asymptotic expansions remain, so these singularities definitely cannot be removed via Schlesinger transformations.

Nevertheless, in cases 3 and 5 one can transform the non-Schlesinger deformations (2.1) to the Schlesinger form (1.1) via the Schlesinger transformations, although in this case one cannot reduce the number of poles in the transformed version of equation (2.18) as can be done in the other cases considered above. Actually, the idea of this proof[1] is quite simple and works for general weak non-Schlesinger deformations of equation (2.18) in $n \times n$ matrices. However, to avoid long introduction of the notation, I will explain it by using the redundant notation, which has already been introduced. Suppose that equation (2.20) defines some weak non-Schlesinger deformation of equation (2.18). Denote by $\{t_1, \ldots, t_m\}$ the set of poles of the function $R(\lambda)$. One of them may coincide with t and simultaneously they belong to the set of the *first-order poles* of equation (2.18). Note that $t_1, \ldots, t_m$ are not supposed to be the first-order poles of $R(\lambda)$. Denote by $C_1(t), \ldots, C_m(t)$ and $\Xi_1, \ldots, \Xi_m$ the matrices defining pole expansions (the matrices like $C(t)$ and Ξ in equations (3.6) and (3.11)) of the solution Ψ at the points $t_1, \ldots, t_m$, respectively. The matrices $C_k(t)$ may also depend on some other parameters, in particular, on t_k, whilst the upper nilpotent matrices Ξ_k are chosen to be constants. The key observation is that the matrices $C_k(t)C_k^{-1}(t_0)$, where $t_0 \notin \{t_1, \ldots, t_m\}$ is any point chosen such that all matrices $C_k(t_0)$ are invertible, commute with Ξ_k. This fact follows from the isomonodromy condition: for any monodromy matrix $M_k(t) = M_k(t_0)$. The inverse monodromy problem at $t = t_0$ is solvable, since it is supposed that the matrices $C_k(t_0)$ exist; therefore, we can construct the Schlesinger deformation with the same monodromy data as for the function Ψ, but with $C_k(t) \to C_k(t_0)$. This fact follows from the solubility of the Cauchy problem at $t = t_0$ for the (generalized) Schlesinger system for the coefficients of equation (2.18). Now, denote by Ψ_S the solution corresponding to the Schlesinger deformations of equation (2.18). Consider analytic properties of $\Psi\Psi_S^{-1}$ as the function of $\lambda \in \mathcal{CP}^1$ and prove (Liouville theorem) that it is a rational function of λ (here the commutativity $C_k(t)C_k^{-1}(t_0)$ with Ξ_k is used). Therefore, Ψ and Ψ_S are related by some Schlesinger transformation.

It is an interesting problem to classify all cases when a weakly non-Schlesinger equation (2.18) in $n \times n$ matrices can be reduced to a simplified equation of the same type which is either independent or 'Schlesinger' with respect to the corresponding parameter(s).

4. Some examples of non-isomonodromic deformations of 3 × 3 matrix ODEs

It is easy to see that any function Ψ which solves equation (2.18) with the rhs given by equation (2.19), where $B(\lambda)$ is a rational function of λ holomorphic at $\lambda = t$, and depends isomonodromically on the parameter t, also satisfies the equation

$$\frac{d\Psi}{dt} = U(\lambda, t)\Psi \tag{4.1}$$

[1] For the Fuchsian systems the idea of the proof was communicated to me by Bolibruch.

where $U(\lambda, t)$ is a rational function of λ. At the same time there are some deformations of equation (2.18) which are defined by equation (4.1) with a non-rational matrix $U(\lambda, t)$, so that these deformations are non-isomonodromic. One interesting example of the non-isomonodromic deformations is considered in [9]. In that example, the non-rational contribution to the matrix $U(\lambda, t)$ is proportional to the identity matrix and therefore reducible to isomonodromic deformations of the Schlesinger type via λ-independent gauge transformations. Here, we consider some simplest examples of the non-isomonodromic deformations for the following 3×3 matrix ODE:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\lambda} = \left(\frac{A_0}{\lambda} + \frac{A}{\lambda - t} + \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & -2\theta \end{pmatrix}\right)\Psi \tag{4.2}$$

where θ is a parameter. Non-isomonodromic deformations considered below are also reducible to the isomonodromic ones; however, these reductions less straightforward as in the example given in [9].

The first deformation is defined by the following ODE:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \left(-\frac{A}{\lambda - t} + P\ln(\lambda - t)\right)\Psi \qquad P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

Denote

$$U = A_0 + A \qquad \Theta = \mathrm{diag}\{\theta, \theta, -2\theta\}$$

then the compatibility condition of equations (4.2) and (4.3) reads

$$PA_0 = A_0 P \qquad PU = UP \qquad P\Theta = \Theta P \tag{4.4}$$

$$A_0' = \frac{1}{t}[U, A_0] \qquad U' = P + [\Theta, U - A_0]. \tag{4.5}$$

The solution of the system (4.4) and (4.5) can be written as follows:

$$A_0 = \begin{pmatrix} a_1(t) & a_1(t) + \theta_1 & a_3(t) \\ a_1(t) + \theta_1 & a_1(t) & a_3(t) \\ a_2(t) & a_2(t) & -2a_1(t) - \theta_2 \end{pmatrix}$$

$$U = \begin{pmatrix} \theta_4 & t + \theta_5 & u_3(t) \\ t + \theta_5 & \theta_4 & u_3(t) \\ u_2(t) & u_2(t) & t + \theta_0 + \theta_4 + \theta_5 \end{pmatrix}$$

where θ_k, $k = 0, \ldots, 5$, are parameters and

$$4a_1(t) + \theta_1 + \theta_2 = \sqrt{\theta_3^2 - 8a_2(t)a_3(t)} \qquad a_3(t) = -\frac{u'(\tau)}{2\sqrt{2}}\mathrm{e}^{\tau} \qquad a_2(t) = \frac{v'(\tau)}{2\sqrt{2}}\mathrm{e}^{-\tau}$$

$$u_3(t) = \frac{u(\tau)}{2\sqrt{2}}\mathrm{e}^{\tau} \qquad u_2(t) = \frac{v(\tau)}{2\sqrt{2}}\mathrm{e}^{-\tau} \qquad \tau = 3\theta t.$$

The functions $u = u(\tau)$ and $v = v(\tau)$ satisfy the following system:

$$\tau u'' = u\sqrt{\theta_3^2 + u'v'} - (\theta_0 + \tau)u' \tag{4.6}$$

$$\tau v'' = v\sqrt{\theta_3^2 + u'v'} + (\theta_0 + \tau)v'. \tag{4.7}$$

This system can be reduced to one ODE of the second order. To prove this let us define auxiliary variables,

$$f = uv \qquad g = \sqrt{\theta_3^2 + u'v'} \qquad h = u'v - v'u.$$

One finds the following equations:

$$\begin{aligned} &f' = 2\tau g' \qquad h' = -2(\theta_0 + \tau)g' \qquad \tau h' = -(\theta_0 + \tau) f' \\ &f'^2 = h^2 + 4(g^2 - \theta_3^2) f \qquad 2\theta_0 g + f + h = \kappa \end{aligned} \tag{4.8}$$

where κ is the constant of integration. By excluding the function h we arrive at the following system of ODEs:

$$\begin{aligned} &\tau((\tau g')' - g^2 + \theta_3^2) + \tfrac{1}{2}(\theta_0 + \tau)(\kappa - 2\theta_0 g) = \tfrac{1}{2}(2g + \theta_0 + \tau) f \\ &(\tau g')^2 = \tfrac{1}{4}(\kappa - 2\theta_0 g - f)^2 + (g^2 - \theta_3^2) f. \end{aligned} \tag{4.9}$$

Finding f from the first equation and substituting it into the second one, we obtain an ODE for the function g which is quadratic with respect to the second derivative. This equation can be solved in terms of the fifth Painlevé transcendent: the following derivation of this fact is due to Cosgrove.

First, observe that f can be eliminated to give the following second-order second-degree differential equation for g:

$$\begin{aligned} &(\tau^2 g'' + \tau g' + 2g^3 + 3\theta_0 g^2 - (\kappa + 2\theta_3^2) g - \theta_0 \theta_3^2)^2 \\ &\qquad = (2g + \tau + \theta_0)^2 (\tau^2 (g')^2 + (g^2 - \theta_3^2)(g^2 + 2\theta_0 g - \kappa - \theta_3^2)). \end{aligned} \tag{4.10}$$

Equations gauge equivalent to (4.10) were known to Chazy [11] and Bureau [12]. The singular integral is immediately apparent: if the last factor on the rhs is set to zero, then the lhs also vanishes and $g(\tau)$ becomes an elliptic function of the variable $\log \tau$.

To obtain the general integral, first observe that g'' can be eliminated between the first equation in (4.9) and the derivative of the second. The result can be factorized as

$$(f' - 2\tau g')(f + 2g^2 + 2\theta_0 g - \kappa - 2\theta_3^2) = 0. \tag{4.11}$$

The second factor yields the singular integral again. The first factor, which will lead to the general integral, vanishes when we express f and g in terms of an auxiliary variable $H(\tau)$ according to

$$f = -4(\tau H' - H) - \theta_0^2 \qquad g = -2H' - \tfrac{1}{2}\theta_0. \tag{4.12}$$

When f and g are eliminated in favour of H, we obtain the following second-order second-degree equation:

$$\tau^2 (H'')^2 = -4(H')^2(\tau H' - H) + A_1(\tau H' - H)^2 + A_2(\tau H' - H) + A_3 H' + A_4 \tag{4.13}$$

where

$$\begin{aligned} &A_1 = 1 \qquad A_2 = \tfrac{1}{4}(3\theta_0^2 + 4\theta_3^2 + 2\kappa) \qquad A_3 = \tfrac{1}{2}\theta_0(\theta_0^2 + \kappa) \\ &A_4 = \tfrac{1}{16}((\theta_0^2 + \kappa)(3\theta_0^2 + \kappa) + 4\theta_0^2\theta_3^2). \end{aligned}$$

Equation (4.13) or a gauge-equivalent version appears in several references, including [10–12]. We have presented it in the standard form of equation (SD-I.b) in [10]. Its general solution is

$$\begin{aligned} H = \frac{1}{4w}\left(\frac{\tau w'}{w-1} - w\right)^2 - \frac{1}{4}(\theta_0^2 + \theta_3^2 + \kappa)(w-1) + \frac{\theta_3^2(w-1)}{4w} \\ + \frac{\theta_0 \tau (w+1)}{4(w-1)} - \frac{\tau^2 w}{4(w-1)^2} \end{aligned} \tag{4.14}$$

where the variable $w(\tau)$ is any solution of the Painlevé-V equation,

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{1}{\tau} w' + \frac{(w-1)^2}{\tau^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{\tau} + \frac{\delta w(w+1)}{w-1} \tag{4.15}$$

having parameters

$$\alpha = \tfrac{1}{2}(1+\sigma)^2 \qquad \text{where} \quad \sigma := \pm\sqrt{\theta_0^2+\theta_3^2+\kappa}$$
$$\beta = -\tfrac{1}{2}\theta_3^2 \qquad \gamma = \theta_0 \qquad \delta = -\tfrac{1}{2}.$$

The Schlesinger and Lukashevich transformations admitted by the Painlevé-V equation induce corresponding symmetries in the f, g-system (4.9). One of these, which is easily found directly, is the involution,

$$f \to f - 2\theta_3(\theta_3+\sigma) - \kappa \qquad g \to g + \tfrac{1}{2}(\theta_0-\theta_3-\sigma) \qquad \sigma \to \tfrac{1}{2}(\theta_0-\theta_3+\sigma)$$
$$\theta_0 \to \theta_3+\sigma \qquad \theta_3 \to \tfrac{1}{2}(\theta_0+\theta_3-\sigma) \qquad \kappa \to (\theta_0-2\theta_3)\sigma - \theta_0^2 - \theta_0\theta_3 - 2\theta_3^2 - \kappa.$$

As is well known, there are several classes of Painlevé-V equations that can be transformed into the Painlevé-III equation,

$$u'' = \frac{(u')^2}{u} - \frac{u'}{\tau} + \tilde{\alpha}u^3 + \frac{\tilde{\beta}}{\tau}u^2 + \frac{\tilde{\gamma}}{\tau} + \frac{\tilde{\delta}}{u}. \tag{4.16}$$

The case $\theta_0 = \kappa = 0$, with θ_3 arbitrary, is one such example. In terms of an auxiliary variable $p(\tau)$, we have

$$f = p^2 \qquad g = \tau p^{-1}(p'' - \tfrac{1}{4}p) \tag{4.17}$$

where p satisfies the second-order second-degree equation,

$$\tau^2\left(p'' - \frac{p}{4}\right)^2 = p^2\left((p')^2 - \frac{p^2}{4} + \theta_3^2\right). \tag{4.18}$$

Equation (4.18) is a special case of an equation appearing in [10–12]. Its solution is given by

$$p = \tau u^{-1}(u' + u^2 - \tfrac{1}{16}) \tag{4.19}$$

where $u(\tau)$ is any solution of the Painlevé-III equation (4.16) with parameters

$$\tilde{\alpha} = 1 \qquad \tilde{\beta} = -(1 \pm 2\theta_3) \qquad \tilde{\gamma} = \tfrac{1}{16}(1 \pm 2\theta_3) \qquad \tilde{\delta} = -\tfrac{1}{256}.$$

The involution $u \to 1/(16u)$ induces the reflection $p \to -p$, which leaves f and g invariant. The above involution admitted by f and g induces a similar Painlevé-III solution of the f, g-system having $\kappa = -\theta_0^2$ and $\theta_3 = 0$.

Of course, we can choose the more general form of the matrix P in these examples; in particular it is clear that everything goes through for $P \longrightarrow \mu(t)P$ where $\mu(t)$ is an arbitrary function. This will lead to a generalization of equation (4.9) which contains the function $\mu(t)$. However, most likely the latter equation will be again equivalent to the fifth Painlevé equation, since as is shown in the work [10] the class of equations quadratic with respect to the second derivatives and solvable in terms of the Painlevé transcendents contains equations with the coefficients depending on some arbitrary functions which are not removable by simple scaling transformations.

An explanation of the appearance of the Painlevé transcendents in the description of the non-isomonodromic deformations is that these deformations preserve some subset of the monodromy data of equation (4.2). More precisely, the transformation $\Psi = Q\tilde{\Psi}$, where the matrix

$$Q = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

diagonalizes P, $Q^{-1}PQ = (1, 1, -1)$, transforms system (4.2), (4.3) to the block form,

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Therefore, denoting $\hat{A}$ and $\hat{A}_0$ 2×2 minors with non-vanishing elements of the matrices $Q^{-1}AQ$ and $Q^{-1}A_0Q$ correspondingly, one finds that vector $\hat{\Psi}$, formed with the first two components of the vector $\tilde{\Psi}$, solves the following system of 2×2 ODEs:

$$\frac{\mathrm{d}\hat{\Psi}}{\mathrm{d}\lambda} = \left(\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\hat{A}_0}{\lambda} + \frac{\hat{A}}{\lambda - t}\right)\hat{\Psi} \qquad \frac{\mathrm{d}\hat{\Psi}}{\mathrm{d}t} = -\frac{\hat{A}}{\lambda - t}\hat{\Psi}. \tag{4.20}$$

As is well known from [5] the monodromy data of fundumental solutions of system (4.20) are independent of t and corresponding isomonodromic deformations of its coefficients are governed by the fifth Painlevé equation. This is another derivation of the relation of the system (4.9) with the fifth Painlevé equation.

Another deformation of equation (4.2) can be defined as follows:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \left(-\frac{A}{\lambda - t} + P_1(\lambda - t)^{\alpha}\right)\Psi \qquad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.21}$$

The compatibility condition of (4.21) with (4.2) reads

$$[\Theta, P_1] = [A_0, P_1] = 0 \qquad [A_1, P_1] = \alpha P_1 \tag{4.22}$$

$$A_0' = \frac{1}{t}[A_1, A_0] \qquad A_1' = \frac{1}{t}[A_0, A_1] + [\Theta, A_1]. \tag{4.23}$$

In the case of 3×3 matrices the system (4.22), (4.23) reduces to the equation for the confluent hypergeometric functions; for higher matrix dimensions it is a system of nonlinear ODEs.

One can consider non-isomonodromic deformations of equation (4.2) more complicated than (4.3) and (4.21) by adding into the rhss of these equations some further terms (such as $\log^2$). I believe that such non-isomonodromic deformations require further study. In particular, a combination of the non-Schlesinger terms, the terms considered in this and the previous sections, could possibly lead to non-Painlevé-type equations.

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